Pasinetti's hyper-integrated labour coefficients and the labour theory of value

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Abstract

Pasinetti (1988) demonstrates that the 'natural' prices of a non-uniformly growing, multisector economy are proportional to "vertically hyper-integrated labour coefficients", a result that constitutes "a complete generalization of the pure labour theory of value". Pasinetti also demonstrates that Marx's "prices of production" cannot be proportional to the hyper-integrated coefficients, a result that constitutes a "complete generalization of Marx's 'transformation problem'". Pasinetti therefore restricts the applicability of the "pure labour theory of value" to a pre-institutional, or 'natural' stage of investigation. In this paper I show that Pasinetti's theoretical innovations, specifically his proposal to extend the classical concept of 'labour embodied' to include additional indirect labour supplied during the period of production, imply that production-prices are proportional to "vertically super-integrated labour coefficients". In consequence, the "pure labour theory of value", suitably generalized, also applies in the institutional circumstances of capitalism, a result that constitutes a general solution to Marx's "transformation problem".

Keywords: Pasinetti, labour theory of value, Marx, transformation problem
JEL: C67, E11, D46
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1. Introduction

Luigi Pasinetti's "separation theorem" (Pasinetti, 2007, Ch. IX) separates the study of economic systems into a "'natural' stage of investigation" and an "institutional stage" (Pasinetti, 2007, p. 276), where the natural stage concerns "the foundational bases of economic relations", and the institutional stage is "carried out at the level of the actual economic institutions" (Pasinetti, 2007, p. 275). Pasinetti's conception of the labour theory of value is shaped by this separation.

For example, Pasinetti (1988) analyzes a non-uniformly growing, multisector economy and proves that the 'pre-institutional' or 'natural' prices, with non-uniform 'natural' profit-rates, are proportional to a generalized measure of labour cost, the "vertically hyper-integrated labour coefficients", which extends the classical concept of "labour-value" to include the labour supplied to produce net investment goods ("hyper-indirect" labour). In consequence, the labour 'embodied' in a commodity, suitably generalized, equals the labour it 'commands' in the market. At this 'natural' stage of investigation, therefore, the "pure labour theory of value" holds Pasinetti (1988, p. 130).

[1]I would like to dedicate this paper to the memory of Angelo Reati. My thanks to Phillip Adkins, Nadia Garbellini, Andrew Trigg and Ariel Luis Wirkierman for feedback and suggestions.

However, at the institutional stage of investigation, Pasinetti argues that the "pure labour theory of value" breaks down. Marx's "prices of production" (Marx, 1971, ch. 9), with a uniform, general rate of profit across all sectors of production, correspond to the natural prices that emerge from an institutional setup in which capitalists reallocate their capital to seek higher returns. Pasinetti proves that, in general, production-prices are not proportional to the hyper-integrated labour coefficients, a result that provides a "complete generalization of Marx's 'transformation problem'" (Pasinetti, 1988, p. 131).

The transformation problem implies that "a theory of value in terms of pure labour can never reflect the price structure that emerges from the operation of the market in a capitalist economy, simply because the market is an institutional mechanism that makes proportionality to physical quantities of labour impossible to realize" (Pasinetti, 1981, p. 153). Pasinetti therefore restricts the "pure labour theory of value" to a foundational, or normative, role, echoing Adam Smith's restriction of the labour theory of value to an "early and rude state of society" that precedes the "accumulation of stock" (Smith, [1776] 1994, p. 53).

Pasinetti's "complete generalization of the pure labour theory of value" (Pasinetti, 1988, p. 130) derives from the theoretical innovation of extending the classical concept of 'labour embodied' to include additional indirect labour supplied during the period of production. In this paper, I further generalize the classical concept and construct the "vertically super-integrated labour coefficients" that additionally include the labour supplied to produce capitalist consumption goods ("super-indirect" labour). I then prove, in the context of Pasinetti's non-uniformly growing economic system, that Marx's production-prices are proportional to the super-coefficients, a result that constitutes a general solution to Marx's transformation problem. In consequence, the "pure labour theory of value" is not restricted to a purely foundational, or normative role, but also applies to the natural price structure of a capitalist economy.

The structure of this paper is as follows: Sections 2 to 5 summarize Pasinetti's model and his argument for restricting the labour theory of value to a 'natural' stage of investigation. Section 6 presents the main argument of this paper and proves that Marx's production-prices are proportional to "vertically super-integrated labour coefficients". Section 7 concludes by discussing the implication of this result for Pasinetti's conception of the labour theory of value.

2. Hyper-subsystems and their natural prices

Sraffa (1960, p. 89) proposed that an integrated economic system can be usefully decomposed into "as many parts as there are commodities in its net product, in such a way that each part forms a smaller self-replacing system the net product of which consists of only one kind of commodity. These parts we shall call 'subsystems'."

Pasinetti (1988) generalizes Sraffa's concept of a subsystem and constructs a n-sector economy, which exhibits unbalanced growth, in terms of n "hyper-subsystems". A hyper-subsystem is a growing, vertically integrated, and therefore self-contained, subsystem dedicated to the production of a single consumption good as net output. The total output of a hyper-subsystem is

$$\mathbf{q}_i(t) = \mathbf{q}_i(t)\mathbf{A}^{\mathrm{T}} + (g + r_i)\mathbf{q}_i(t)\mathbf{A}^{\mathrm{T}} + \mathbf{n}_i(t), \tag{1}$$

where $\mathbf{q}_i(t)$ is a vector of n quantities, $\mathbf{A} = [a_{i,j}]$ is a constant $n \times n$ input-output matrix, and $\mathbf{n}_i(t)$ is a zero vector except for the ith component, which is a scalar n_i that represents the final demand for commodity i. The total output, $\mathbf{q}_i(t)$, breaks down into (i) replacement for used-up means of production, $\mathbf{q}_i(t)\mathbf{A}^{\mathrm{T}}$, (ii) additional investment in means of production, $(g + r_i)\mathbf{q}_i(t)\mathbf{A}^{\mathrm{T}}$, to meet increased demand for commodity i due to the growth rate, g, of the population and the per-capita growth rate, r_i, of consumption demand for commodity i (which may be positive or negative), and (iv) the final output, or net product, $\mathbf{n}_i(t)$, which is the quantity of commodity i consumed. The total labour supplied to the hyper-subsystem is

$$L_i(t) = \mathbf{l}\mathbf{q}_i^{\mathrm{T}}(t) = \mathbf{l}\,(\mathbf{I} - \mathbf{A}(1 + g + r_i))^{-1}\,\mathbf{n}_i^{\mathrm{T}}(t), \tag{2}$$

where $\mathbf{l} = [l_i]$ is a vector of n direct labour coefficients.

A hyper-subsystem includes the labour and means of production necessary for the production of its output *and* the labour and net investment in means of production necessary for its expansion at the growth rate $(g + r_i)$.[2] Pasinetti defines the trajectory of final demand as defines as

$$n_i(t) = n_i(0)\mathrm{e}^{(g+r_i)t} \tag{3}$$

(i.e., $\frac{\mathrm{d}n_i}{\mathrm{d}t} = n_i(g + r_i)$), which drives the growth of the subsystem.

For notational convenience I now drop explicit time parameters. All subsequent algebraic statements therefore hold at an implicit time t. (I consider the implications of the trajectory of final demand in the appendix).

Pasinetti (1989) defines the gross output of the integrated economic system, which exhibits non-uniform growth, in terms of its n hyper-subsystems,

$$\mathbf{q} = [q_i] = \mathbf{q}\mathbf{A}^{\mathrm{T}} + g\mathbf{q}\mathbf{A}^{\mathrm{T}} + \left(\sum_{i=1}^{n} r_i \mathbf{q}_i\right)\mathbf{A}^{\mathrm{T}} + \mathbf{n}, \tag{4}$$

where $\mathbf{q} = \sum_{i=1}^{n} \mathbf{q}_i$, $\mathbf{n} = \sum_{i=1}^{n} \mathbf{n}_i$, and $L = \mathbf{l}\mathbf{q} = \sum_{i=1}^{n} L_i$, with standard restrictions on the eigenvalues of $\mathbf{A}$ and the feasibility of growth rates.[3]

Pasinetti then defines natural prices for this economic system that correspond to a 'natural', or pre-institutional, stage of investigation. He stipulates that each hyper-subsystem has its own natural profit-rate, $\pi_i^\star$, which "is equal to the rate of growth of demand for the corresponding consumption good" (Pasinetti, 1988, p. 129); that is, we have n natural profit-rates, $\pi_1^\star, \pi_2^\star, \ldots, \pi_n^\star$, where $\pi_i^\star = g + r_i$, and, in consequence, n vectors of natural prices, $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$, one for for each hyper-subsystem, such that

$$\mathbf{p}_i = \mathbf{p}_i\mathbf{A} + \mathbf{p}_i\mathbf{A}\pi_i^\star + \mathbf{l}w, \tag{5}$$

where w is the wage rate.

Define $\mathbf{X}^{(i)}$ as the ith column and $\mathbf{X}_{(i)}$ as the ith row of $\mathbf{X}$. The price of commodity j in hyper-subsystem i therefore breaks down into (i) the cost of replacing used-up means of production, $\mathbf{p}_i\mathbf{A}^{(j)}$, (ii) the cost of net investment in additional means of production, $\mathbf{p}_i\mathbf{A}^{(j)}\pi_i^\star$, and (iii) the wage bill, $l_i w$.

[2]Equations (1) and (2) are identical to equations (2.5), (2.6), and (2.7) in (Pasinetti, 1988) except, in this paper, we make the simplifying assumption that $\mathbf{B} = \mathbf{I}$.

[3]Equation (4) is identical to equation (2.1) in (Pasinetti, 1989), except, again, we set $\mathbf{B} = \mathbf{I}$.

In general, each commodity-type has a different natural price in each hyper-subsystem. The natural profit-rates "make possible the expansion of the production of each final good according to the evolution of its final demand", which "provides a 'social' justification for profit" (Bellino, 2009) in terms of 'mark-up' rates objectively determined by technology and growth.

3. "A complete generalization of the pure labour theory of value"

Pasinetti relates the 'natural' prices to a specific, and novel, measure of labour costs.

Classical labour-values measure the direct and indirect labour supplied to produce unit commodities. The standard equation for classical labour-values, $\mathbf{v} = \mathbf{vA} + \mathbf{l}$, sums the direct labour ($\mathbf{l}$) with the indirect labour ($\mathbf{vA}$) used-up to replace means of production (e.g., see Sraffa (1960), Samuelson (1971) and Pasinetti (1977)).

The natural prices, $\mathbf{p}_i$, of a single hyper-subsystem, given by equation (5), are a function of the natural profit-rate, $\pi_i^\star = g + r_i$, whereas classical labour-values are not. The natural prices vary independently of classical labour-values and cannot be reduced to the direct and indirect labour supplied to produce commodities.

Pasinetti has therefore demonstrated that capitalist profit is not the essential cause of the divergence of natural prices and classical labour-values. In the economy defined by (4), with the set of natural prices (5), wages are the only type of income. Capitalist profit, in the sense of income received in virtue of firm ownership rather than labour supplied, is absent at this 'natural' stage of investigation. As Reati (2000) notes, "the mere existence of wages could presuppose two social classes. However, on this point also Pasinetti's model is flexible, because nothing prevents us from considering a self-managed economy in which workers decide on the amount and allocation of a surplus". In consequence, even in the absence of capitalist profit, a "transformation problem" arises: the natural prices cannot be reduced to "labour-values". Capitalist social relations are therefore merely a sufficient, not a necessary, condition for the divergence of natural prices and labour-values (this point was made, somewhat differently, by von Weizsäcker and Samuelson (1971)).

However, Pasinetti's analysis demonstrates that classical labour-values do not exhaust the possible measures of labour cost. Pasinetti constructs "vertically hyper-integrated labour coefficients" that generalize the classical definition to include the labour supplied to produce net investment goods:

Definition 1. *Pasinetti's vertically hyper-integrated labour coefficients, $\mathbf{v}^\star$, are*

$$\mathbf{v}_i^\star = \mathbf{l} + \mathbf{v}_i^\star\mathbf{A} + \mathbf{v}_i^\star\mathbf{A}(g + r_i), \tag{6}$$

which is the sum of direct, indirect and "hyper-indirect" labour.[4]

In conditions of zero growth, i.e. $g + r_i = 0$, the hyper-integrated labour coefficients reduce to classical labour-values, i.e. $\mathbf{v}_i = \mathbf{l} + \mathbf{v}_i\mathbf{A}$.

Pasinetti then demonstrates that the natural prices in each hyper-subsystem are proportional to the vertically hyper-integrated labour coefficients:

$$\mathbf{p}_i = \mathbf{p}_i\mathbf{A} + \mathbf{p}_i\mathbf{A}\pi_i^\star + \mathbf{l}w = \mathbf{l}\left(\mathbf{I} - \mathbf{A}(1 + \pi_i^\star)\right)^{-1} w = \mathbf{v}_i^\star w.$$

[4]Equation (6) is identical to equation (2.9) in (Pasinetti, 1988) except $\mathbf{B} = \mathbf{I}$.

'Natural' prices, therefore, are the total wage bill of the direct, indirect and hyper-indirect labour supplied to produce unit commodities.

Pasinetti (1988, p. 130) notes "this is a complete generalization of the pure labour theory of value" that recreates Smith's "early and rude state" of society in which labour-embodied equals labour-commanded. Furthermore, "the analytical step that allows the achievement of this result is of course a re-definition of the concept of 'labour embodied', which must be intended as the quantity of labour required directly, indirectly *and* hyper-indirectly to obtain the corresponding commodity as a consumption good" (Pasinetti, 1988, pp. 131–132).

In summary, once we measure the total labour costs of production, i.e. the vertically hyper-integrated labour coefficients, rather than partial labour costs, i.e. classical labour-values, then a "transformation problem" does not arise within the hyper-subsystems.

4. "A complete generalization of Marx's 'transformation problem'"

Pasinetti now switches to an institutional stage of investigation where capitalists, as residual claimants on firm incomes, reallocate their capital seeking the highest returns. At the natural price equilibrium a general, uniform profit-rate prevails across all sectors of the economy; i.e.,

$$\mathbf{p} = \mathbf{pA} + \mathbf{pA}\pi + \mathbf{l}w. \tag{7}$$

Marx (1971, ch. 9) named these prices the "prices of production". Production-prices (7), in contrast to natural prices (5), impose a single price structure on the integrated economy as a whole. Profit, at this institutional stage, is a distributional variable proportional to the investment in means of production within each sector of production, i.e. $\mathbf{pA}\pi$, rather than a set of structural 'mark ups' within each vertically hyper-integrated subsystem, i.e. $\mathbf{p}_i\mathbf{A}\pi_i^\star$.

Pasinetti demonstrates that the proportional relationship between 'natural' prices and hyper-integrated labour coefficients breaks down in the institutional circumstances of capitalism. Pasinetti writes equation (7) in the equivalent form,

$$\mathbf{p} = \mathbf{pA} + \mathbf{pA}(g + r_i) + \mathbf{pA}(\pi - g - r_i) + \mathbf{l}w, \tag{8}$$

where $g + r_i$ is the growth rate of demand for any consumption good we care to choose (here we have chosen the ith commodity). For convenience define the matrix $\mathbf{M}_i = \mathbf{A}\,(\mathbf{I} - \mathbf{A}(1 + g + r_i))^{-1}$. We can therefore write production-price equation (7) in n different, but equivalent, forms,

$$\mathbf{p} = \mathbf{v}_i^\star\,(\mathbf{I} - \mathbf{M}_i(\pi - g - r_i))^{-1}\,w, \qquad i = 1, 2, \ldots, n.^5 \tag{9}$$

[5] Equation (9) is identical to equation (4.5) in Pasinetti (1988), and we derive it from equation (8):

$$\begin{aligned}
\mathbf{p} &= \mathbf{pA} + \mathbf{pA}(g + r_i) + \mathbf{pA}(\pi - g - r_i) + \mathbf{l}w \\
\mathbf{p}(\mathbf{I} - \mathbf{A}(1 + g + r_i)) &= \mathbf{l}w + \mathbf{pA}(\pi - g - r_i) \\
\mathbf{p} &= \mathbf{l}(\mathbf{I} - \mathbf{A}(1 + g + r_i))^{-1}w + \mathbf{pA}(\mathbf{I} - \mathbf{A}(1 + g + r_i))^{-1}(\pi - g - r_i) \\
&= \mathbf{v}_i^\star w + \mathbf{pM}_i(\pi - g - r_i).
\end{aligned}$$

Consider, for a moment, the special, or accidental case, in which the general profit-rate equals the growth rate of demand for commodity i; that is, $\pi = g + r_i$. Equation (9) then collapses to $\mathbf{p} = \mathbf{v}_i^\star w$ and production-prices are proportional to the vertically hyper-integrated labour coefficients of hyper-subsystem i (and therefore commensurate to the 'natural' prices of subsystem i). But in general, $\pi \neq g + r_i$ for any i, and therefore production-prices are not proportional to any hyper-integrated labour coefficient of any hyper-subsystem.

In fact, production-prices vary independently of the vertically hyper-integrated labour coefficients because prices are a function of a global distributional variable, π, whereas the hyper-integrated labour coefficients are not. Production-prices cannot be reduced to labour costs, whether measured in terms of classical labour-values or Pasinetti's hyper-integrated coefficients. Pasinetti (1988, p. 131) notes, therefore, that equation (9) "can also be regarded as providing a complete generalization of Marx's 'transformation problem"' to the case of a non-uniformly growing economy.

5. The pure labour theory of value as a "logical frame of reference"

To summarize: if we relate classical labour-values to the natural prices of a hyper-subsystem we encounter a transformation problem: labour costs and nominal costs are incommensurate. The fundamental reason for the transformation problem is simple: the natural prices of a hyper-subsystem include the cost of net investment as a component of the price of commodities, whereas classical labour-values exclude the labour cost of net investment as a component of the labour-value commodities. In this case, the dual systems of prices and labour-values adopt different, and incommensurate, cost accounting conventions.

This transformation problem dissolves once we relate natural prices to the hyper-integrated labour coefficients. Hyper-integrated labour coefficients adopt the same accounting convention as the price system, and therefore include the labour cost of net investment as a component of the labour-value of commodities. Commensurability is thereby restored.

Pasinetti understands that, in certain circumstances, classical labour-values under-count the total labour costs of production. He therefore constructs a more general measure of labour cost appropriate to the more general economic setting, specifically the vertically hyper-integrated labour coefficients, which include the additional labour supplied to expand a hyper-subsystem at a given growth rate.

However, Pasinetti re-encounters a transformation problem when he relates the hyper-integrated labour coefficients to production-prices with a uniform profit-rate. In an earlier work he concludes that this "analysis amounts to a demonstration that a theory of value in terms of pure labour can never reflect the price structure that emerges from the operation of the market in a capitalist economy, simply because the market is an institutional mechanism that makes proportionality to physical quantities of labour impossible to realize" (Pasinetti, 1981, p. 153).

In consequence, Pasinetti restricts the "pure labour theory of value" to a 'natural' or normative stage of investigation, where it "has to be taken as providing a *logical frame of reference* – a *conceptual construction* which defines a series, actually a family of series, of ideal *natural* prices, which possess an extraordinarily high number of remarkable, analytical, and *normative*, properties" (my emphasis) (Pasinetti, 1988, p. 132).

Is this conclusion correct? I will now show that Pasinetti's theoretical innovations point to the opposite conclusion, specifically that production-prices are proportional to a more general measure of labour cost, the "vertically super-integrated labour coefficients", which measure total labour costs in the institutional circumstances of capitalism. Pasinetti's restriction of the pure labour theory of value to a 'natural' stage of investigation is therefore unwarranted. In consequence, the labour theory of value, suitably generalized, equally applies to the "operation of the market in a capitalist economy".

6. Production-prices proportional to "physical quantities of labour"

I develop the argument by first considering the simpler case of a steady-state economy before generalizing to the full complexity of Pasinetti's growth model.

6.1. A special case: the steady-state economy

Production-prices specify a distribution of nominal income in circumstances where capitalists receive profit income and workers receive wages. In order to define the "vertically super-integrated labour coefficients" we require the corresponding physical data that specifies the distribution of real income. Assume, therefore, that workers receive the real wage, $\mathbf{w} = [w_i]$, and capitalists receive the consumption bundle, $\mathbf{c} = [c_i]$, such that the net product $\mathbf{n} = \mathbf{w} + \mathbf{c}$.

In conditions of zero growth, i.e. $g = 0$ and $r_i = 0$ for all i, Pasinetti's quantity eqauation (4) reduces to $\mathbf{q} = \mathbf{q}\mathbf{A}^{\mathrm{T}} + \mathbf{n}$, which we expand as

$$\mathbf{q} = \mathbf{q}\mathbf{A}^{\mathrm{T}} + \mathbf{w} + \mathbf{c}. \tag{10}$$

We analyze the following special-case, steady-state economy:

Definition 2. *A "steady-state economy with production-prices" produces quantities,* $\mathbf{q} = \mathbf{q}\mathbf{A}^T + \mathbf{w} + \mathbf{c}$*, at prices,* $\mathbf{p} = \mathbf{p}\mathbf{A}(1+\pi) + \mathbf{l}w$*, where workers and capitalists spend what they earn,* $\mathbf{p}\mathbf{w}^T = \mathbf{l}\mathbf{q}^T w$ *and* $\mathbf{p}\mathbf{c}^T = \mathbf{p}\mathbf{A}\mathbf{q}^T\pi$*.*

In this economy the net product is produced, distributed and consumed within the period of production. Over multiple periods the economy self-replaces with a constant composition and scale (i.e., Pasinetti's economy, in conditions of zero growth, reduces to a closed Leontief system with final demand equal to the consumption of workers and capitalists; see Pasinetti (1977, pp. 60–61)).

The production and distribution of the net product are necessarily related. For example, the quantity of commodity i consumed by workers per unit of wage income is $w_i/\mathbf{l}\mathbf{q}^{\mathrm{T}}w$. The wage income received, per unit output in sector j, is $l_j w$. Hence, consumption coefficient $w_{i,j} = w_i l_j/\mathbf{l}\mathbf{q}^{\mathrm{T}}$ denotes the quantity of commodity i distributed to workers per unit output of j. Define

$$\mathbf{W} = \frac{1}{\mathbf{l}\mathbf{q}^{\mathrm{T}}}\mathbf{w}^{\mathrm{T}}\mathbf{l} = [w_{i,j}],$$

as a matrix of worker consumption coefficients. $\mathbf{W}$ compactly describes the physical flow rate of consumption goods to worker households per unit outputs.

Production-price equation (7) implies that profit is proportional to the money-capital 'tied up' in circulating capital, i.e. $\mathbf{p}\mathbf{A}\mathbf{q}^{\mathrm{T}}\pi$. The quantity of commodity i consumed by capitalists per unit of profit income is therefore $c_i/\mathbf{pAq}^{\mathrm{T}}\pi$. The profit income recieved, per unit output in sector j, is $\mathbf{pA}^{(j)}\pi$. Hence, consumption coefficient $c_{i,j} = c_i\mathbf{pA}^{(j)}/\mathbf{pAq}^{\mathrm{T}}$ denotes the quantity of commodity i distributed to capitalists per unit output of j. Define

$$\mathbf{C} = \frac{1}{\mathbf{pAq}^{\mathrm{T}}}\mathbf{c}^{\mathrm{T}}\mathbf{pA} = [c_{i,j}], \tag{11}$$

as a matrix of capitalist consumption coefficients. $\mathbf{C}$ compactly describes the physical flow rate of consumption goods to capitalist households per unit outputs.

Matrices $\mathbf{W}$ and $\mathbf{C}$ synchronize production with the distribution of the net product. Both matrices specify relative material flows of commodities; for example, the unit of measurement of each quantity $w_{i,j}$ or $c_{i,j}$ is identical to the unit of measurement of the corresponding element $a_{i,j}$ in the technique $\mathbf{A}$. In consequence, matrices $\mathbf{A}$, $\mathbf{W}$ and $\mathbf{C}$ are all 'physical' input-output matrices that may be directly observed by measuring flows of goods between sectors of production and households.

A Sraffian subsystem is the direct and indirect production that outputs a single component of the net product. A "vertically super-integrated subsystem", in contrast, is the direct, indirect and "super-indirect" production that outputs a single component of the real wage, where "super-indirect" refers to the production of capitalist consumption goods. The total output of the ith vertically super-integrated subystem is

$$\hat{\mathbf{q}}_i = \hat{\mathbf{q}}_i\mathbf{A}^{\mathrm{T}} + \hat{\mathbf{q}}_i\mathbf{C}^{\mathrm{T}} + \mathbf{w}_i,$$

where $\mathbf{w}_i$ is a zero vector except for the ith component that equals w_i, which is the wage demand for commodity i. The gross output of the steady-state economy is the sum of the output of each vertically super-integrated subsystem, i.e. $\mathbf{q} = \sum \hat{\mathbf{q}}_i$.

A "vertically super-integrated labour coefficient", denoted $\hat{v}_i$, is the total labour supplied to the ith super-integrated subsystem when it outputs a unit component of the real wage, i.e. when $w_i = 1$. For clarity, I will calculate this quantity step-by-step.

Consider the production of 1 unit of commodity i in super-integrated subsystem i. How much labour does this production require? It requires l_i units of direct labour, $\mathbf{lA}^{(i)}$ units of indirect labour, and $\mathbf{lC}^{(i)}$ units of super-indirect labour, giving a total of $\mathbf{l}(\mathbf{A}^{(i)}+\mathbf{C}^{(i)})$ units of labour operating in parallel to produce the output, replace used-up means of production and replace capitalist consumption goods, respectively. Define $\tilde{\mathbf{A}} = \mathbf{A} + \mathbf{C}$ as the technique augmented by capitalist consumption. Matrix $\tilde{\mathbf{A}}$ compactly represents the commodities used-up during the production of each commodity-type *including* the commodities consumed by capitalists. The sum of direct, indirect and super-indirect labour is then $l_i + \mathbf{l}\hat{\mathbf{A}}^{(i)}$.

However, the indirect and super-indirect production itself uses-up means of production and consumption goods, specifically the bundle $\tilde{\mathbf{A}}\tilde{\mathbf{A}}^{(i)}$, which is contemporaneously replaced by the supply of additional labour, $\mathbf{l}\tilde{\mathbf{A}}\hat{\mathbf{A}}^{(i)}$. To count all the direct, indirect and super-indirect labour we must continue the sum; that is,

$$\begin{aligned}\hat{v}_i &= l_i + \mathbf{l}\tilde{\mathbf{A}}^{(i)} + \mathbf{l}\tilde{\mathbf{A}}\tilde{\mathbf{A}}^{(i)} + \mathbf{l}\tilde{\mathbf{A}}^2\tilde{\mathbf{A}}^{(i)} + \ldots \\ &= l_i + \mathbf{l}\left(\sum_{n=0}^{\infty}\tilde{\mathbf{A}}^n\right)\tilde{\mathbf{A}}^{(i)}.\end{aligned}$$

This sum represents the total labour supplied to the ith super-integrated subsystem when it produces unit output.

The vector $\tilde{\mathbf{v}}$ of super-integrated coefficients is therefore $\tilde{\mathbf{v}} = \mathbf{l} + \mathbf{l}\left(\sum_{n=0}^{\infty}\tilde{\mathbf{A}}^n\right)\tilde{\mathbf{A}} = \mathbf{l}\sum_{n=0}^{\infty}\tilde{\mathbf{A}}^n$. On assumption that capitalist consumption is feasible, given the technology, then matrix $\tilde{\mathbf{A}}$ is productive, and we may replace the infinite series with the Leontief inverse, $\mathbf{l}\sum_{n=0}^{\infty}\tilde{\mathbf{A}}^n = \mathbf{l}(\mathbf{I}-\tilde{\mathbf{A}})^{-1}$; in consequence:

Definition 3. *The "vertically super-integrated labour coefficients", $\tilde{\mathbf{v}}$, in a steady-state economy with production-prices, are*

$$\begin{aligned}\tilde{\mathbf{v}} &= \mathbf{l} + \tilde{\mathbf{v}}\tilde{\mathbf{A}} \\ &= \mathbf{l} + \tilde{\mathbf{v}}\mathbf{A} + \tilde{\mathbf{v}}\mathbf{C}, \end{aligned} \tag{12}$$

which is the sum of direct, indirect and super-indirect labour costs.[6]

The definition of the super-integrated coefficients does not provide or rely upon any theory of income distribution or profit. However, in order to calculate the super-integrated coefficients the distribution of real income must be a given datum, in the same manner that, in order to calculate production-prices, the distribution of nominal income must be a given datum.

The super-integrated labour coefficients, although more complex than classical labour-values, nonetheless directly relate to the labour supplied during the production period.

For example, Pasinetti (1980, p. 21) classifies the total labour supplied in two ways: as (i) the sum of direct labour supplied to each sector of production, $\sum l_i q_i = \mathbf{l}\mathbf{q}^{\mathrm{T}}$, or (ii) the sum of direct and indirect labour supplied to each Sraffian subsystem, $\sum v_i n_i = \mathbf{v}\mathbf{n}^{\mathrm{T}}$. The classifications are quantitatively equal, that is $\mathbf{l}\mathbf{q}^{\mathrm{T}} = \mathbf{v}\mathbf{n}^{\mathrm{T}}$, because the Sraffian subsystems collectively output the net product and exhaust the total supplied labour:

Proposition 1. *The total labour supplied equals the classical labour-value of the net product,* $\mathbf{l}\mathbf{q}^T = \mathbf{v}\mathbf{n}^T$.

Proof. From (10), $\mathbf{q} = \mathbf{q}\mathbf{A}^{\mathrm{T}} + \mathbf{n} = \mathbf{n}(\mathbf{I}-\mathbf{A}^{\mathrm{T}})^{-1}$. Hence $\mathbf{l}\mathbf{q}^{\mathrm{T}} = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{n}^{\mathrm{T}} = \mathbf{v}\mathbf{n}^{\mathrm{T}}$. □

The vertically super-integrated sectors provide another partition of the economy. The total labour supplied can also be classified as (iii) the sum of direct, indirect and super-indirect labour supplied to each super-integrated sector, $\sum \hat{v}_i w_i = \hat{\mathbf{v}}\mathbf{w}^{\mathrm{T}}$. Again, this classification is quantitatively equal to the total labour supplied, that is $\mathbf{l}\mathbf{q}^{\mathrm{T}} = \hat{\mathbf{v}}\mathbf{w}^{\mathrm{T}}$, because the super-integrated subsystems collectively output the real wage and exhaust the total supplied labour:

Proposition 2. *The total labour supplied equals the super-integrated labour-value of the real wage,* $\hat{\mathbf{v}}\mathbf{w}^T = \mathbf{l}\mathbf{q}^T$.

Proof. From (11), $\mathbf{C}\mathbf{q}^{\mathrm{T}} = (1/\mathbf{p}\mathbf{A}\mathbf{q}^{\mathrm{T}})\mathbf{c}^{\mathrm{T}}\mathbf{p}\mathbf{A}\mathbf{q}^{\mathrm{T}} = \mathbf{c}^{\mathrm{T}}$. Substitute into (10) to yield, $\mathbf{q} = \mathbf{q}\mathbf{A}^{\mathrm{T}} + \mathbf{q}\mathbf{C}^{\mathrm{T}} + \mathbf{w} = \mathbf{q}\hat{\mathbf{A}} + \mathbf{w} = \mathbf{w}(\mathbf{I}-\hat{\mathbf{A}})^{-1}$. Hence $\mathbf{l}\mathbf{q}^{\mathrm{T}} = (\mathbf{I}-\hat{\mathbf{A}})^{-1}\mathbf{w}^{\mathrm{T}} = \hat{\mathbf{v}}\mathbf{w}^{\mathrm{T}}$. □

Now that we've defined the super-integrated coefficients we can relate them to production-prices. Recall that Marx's transformation problem arises because production-prices vary independently of classical labour-values and therefore cannot be reduced to them. In contrast, production-prices are proportional, and therefore vary one-to-one, with the super-integrated labour coefficients:

[6] I have previously called these coefficients 'nonstandard labour-values' (Wright, 2009, 2014).

Theorem 1. *The production-prices of a steady-state economy are proportional to the super-integrated labour coefficients,* $\mathbf{p} = \tilde{\mathbf{v}}w$.

Proof. Since capitalists spend what they earn, $\mathbf{pAq}^{\mathrm{T}}\pi = \mathbf{pc}^{\mathrm{T}}$. Substitute for π into price equation (7): $\mathbf{p} = \mathbf{pA}(1 + \frac{\mathbf{pc}^{\mathrm{T}}}{\mathbf{pAq}^{\mathrm{T}}}) + \mathbf{l}w = \mathbf{pA} + \frac{\mathbf{pc}^{\mathrm{T}}}{\mathbf{pAq}^{\mathrm{T}}}\mathbf{pA} + \mathbf{l}w = \mathbf{p}(\mathbf{A} + \frac{1}{\mathbf{pAq}^{\mathrm{T}}}\mathbf{c}^{\mathrm{T}}\mathbf{pA}) + \mathbf{l}w = \mathbf{pA} + \mathbf{pC} + \mathbf{l}w = \mathbf{p}\tilde{\mathbf{A}} + \mathbf{l}w = \mathbf{l}(\mathbf{I} - \tilde{\mathbf{A}})^{-1}w = \tilde{\mathbf{v}}w$. □

In consequence, production-prices are the total wage bill of the direct, indirect and super-indirect labour supplied to produce unit commodities. The more general definition of labour costs recreates Adam Smith's "early and rude state" of society in which 'labour embodied' equals 'labour commanded'.

The 'physical' configuration of the steady-state economy, specifically the prevailing technique and distribution of real income, determine both the structure of the vertically super-integrated labour coefficients and the structure of production-prices. In consequence, production-prices and labour costs, suitably measured, are necessarily dual to each other and "two sides of the same coin".

This result, it should be emphasized, destroys the basis of the claim that the labour theory of value is necessarily incoherent because production-prices and labour-values are quantitatively incommensurate in linear production models (e.g., Samuelson (1971); Lippi (1979); Steedman (1981)).

Next I generalize this analysis to Pasinetti's growth model. The generalization does not require any new arguments or ideas. However, in the case of non-uniform growth the super-integrated coefficients include both hyper and super-indirect labour.

6.2. The general case: Pasinetti's non-uniform growth model

Final demand, in circumstances of non-uniform growth, is variable, and therefore the net product is a function of time, i.e. $\mathbf{n}(t) = \mathbf{w}(t) + \mathbf{c}(t)$. We therefore assume an initial distribution of real income, $\mathbf{w}(0)$ and $\mathbf{c}(0)$. The trajectory of final demand, from equation (3), is then $\mathbf{n}(t) = [w_i(0)\mathrm{e}^{(g+r_i)t}] + [c_i(0)\mathrm{e}^{(g+r_i)t}]$. Vectors $\mathbf{w}$ and $\mathbf{c}$ now implicitly refer to time-varying consumption bundles that, following Pasinetti, drive the growth of the economy.[7]

For notational convenience define the "non-uniform capital investment vector", $\mathbf{g} = \sum_{i=1}^{n} r_i\mathbf{q}_i\mathbf{A}^{\mathrm{T}} = [g_i]$, where $\mathbf{q}_i = [q_{i,j}]$ is the gross output of hyper-subsystem i as defined by equation (1), and each g_i is the quantity of commodity i produced as additional means of production, in the economy as a whole, in order to meet the total non-uniformly growing demand.[8] Let $\mathbf{\Gamma} = \mathbf{diag(g)}\ \mathbf{diag(q)}^{-1} = [\lambda_{i,j}]$ be a diagonal "non-uniform capital investment matrix", where each element on the diagonal, $\lambda_{i,i} = g_i/q_i$, is the quantity of i produced as additional means of production, per unit output, to meet the total non-uniformly growing demand (otherwise $\lambda_{i,j} = 0$ for $i \neq j$). We can therefore rewrite Pasinetti's quantity equation (4) in the equivalent form,

$$\mathbf{q} = \mathbf{qA}^{\mathrm{T}}(1 + g) + \mathbf{q\Gamma} + \mathbf{w} + \mathbf{c}. \tag{13}$$

[7] Of course, in the context of an actual capitalist economy, rather than Pasinetti's system, growth is not driven by exogenous real demand. Our goal is to explore the full consequences of Pasinetti's imposition of a single 'capitalist' price structure on his model rather than develop a realistic growth model.

[8] Vector **g** features in Pasinetti's equation (4) that defines the gross ouput of the integrated economy.

As before the total profit income is $\mathbf{pAq}^{\mathrm{T}}\pi$. But now a fraction of profit is invested in additional means of production to satisfy increased demand: $\mathbf{pAq}^{\mathrm{T}}g$ is invested to satisfy the increase in demand due to population growth and $\mathbf{p\Gamma q}^{\mathrm{T}}$ is invested to satisfy the non-uniform change in demand. The residual profit that remains for capitalists to spend on personal consumption is therefore $Y = \mathbf{p}(\mathbf{A}(\pi - g) - \mathbf{\Gamma})\mathbf{q}^{\mathrm{T}}$.

The full specification of Pasinetti's non-uniformly growing economy with production-prices is therefore:

Definition 4. *A "non-uniformly growing economy with production-prices" produces quantities,* $\mathbf{q} = \mathbf{qA}^T(1 + g) + \mathbf{q\Gamma} + \mathbf{w} + \mathbf{c}$, *at prices,* $\mathbf{p} = \mathbf{pA}(1 + \pi) + \mathbf{l}w$, *where workers and capitalists spend what they earn,* $\mathbf{pw}^T = \mathbf{lq}^T w$ *and* $\mathbf{pc}^T = \mathbf{p}(\mathbf{A}(\pi - g) - \mathbf{\Gamma})\mathbf{q}^T$.

The production and distribution of the net product are, once again, necessarily related. The matrix of worker consumption coefficients, $\mathbf{W}$, is unchanged from the steady-state case. However, the matrix of capitalist consumption coefficients, $\mathbf{C}$, differs because capitalists invest a fraction of their profit income in additional means of production. The quantity of commodity i consumed by capitalists per unit of residual profit income is now c_i/Y. The residual profit recieved, per unit output in sector j, is $\mathbf{p}(\mathbf{A}^{(j)}(\pi - g) - \mathbf{\Gamma}^{(j)})$. Hence consumption coefficient $c_{i,j} = c_i\mathbf{p}(\mathbf{A}^{(j)}(\pi - g) - \mathbf{\Gamma}^{(j)})/Y$ denotes the quantity of commodity i distributed to capitalists per unit output of j. Hence

$$\mathbf{C} = \frac{1}{\mathbf{p}(\mathbf{A}(\pi - g) - \mathbf{\Gamma})\mathbf{q}^{\mathrm{T}}}\mathbf{c}^{\mathrm{T}}\mathbf{p}(\mathbf{A}(\pi - g) - \mathbf{\Gamma}) = [c_{i,j}] \tag{14}$$

is the matrix of capitalist consumption coefficients. (Note that, when $g = 0$ and $r_i = 0$ for all i, this definition of $\mathbf{C}$ reduces to the definition for the steady-state economy).

A Pasinettian hyper-integrated subsystem is the direct, indirect and hyper-indirect production that outputs a single component of the net product. A vertically super-integrated subsystem, in the context of non-uniform growth, is the direct, indirect, hyper and super-indirect production that outputs a single component of the real wage. The total output of the ith vertically-super integrated subsystem is

$$\hat{\mathbf{q}}_i = \hat{\mathbf{q}}_i\mathbf{A}^{\mathrm{T}} + \hat{\mathbf{q}}_i\mathbf{A}^{\mathrm{T}}g + \hat{\mathbf{q}}_i\mathbf{\Gamma} + \hat{\mathbf{q}}_i\mathbf{C}^{\mathrm{T}} + \mathbf{w}_i,$$

where $\mathbf{w}_i$ is defined as before and $\mathbf{q} = \sum \hat{\mathbf{q}}_i$.

The production of net investment in a hyper-integrated subsystem is, from equation (1), $\mathbf{q}_i\mathbf{A}^{\mathrm{T}}(g + r_i)$, which is independent of the cross-demand effects of the non-uniform growth of the other hyper-integrated subsystems (i.e., r_j for all $j \neq i$). In contrast, the production of net investment in a super-integrated subsystem, $\hat{\mathbf{q}}_i\mathbf{A}^{\mathrm{T}}g + \hat{\mathbf{q}}_i\mathbf{\Gamma}$, includes cross-demand effects. In other words, the super-integrated subsystems, unlike the hyper-integrated subsystems, are not 'natural' or pre-institutional structures. The super-integrated subsystems partition Pasinetti's economy according to the specific institutional circumstances of production organized by a capitalist class.

The technique augmented by hyper and super-indirect real costs of production is then

$$\hat{\mathbf{A}} = \mathbf{A} + \mathbf{A}g + \mathbf{\Gamma} + \mathbf{C}.$$

Matrix $\hat{\mathbf{A}}$ compactly represents the commodities used-up during the production of each commodity-type including the production of net investment goods and capitalist consumpion goods.

Definition 5. *The "vertically super-integrated labour coefficients",* $\hat{\mathbf{v}}$*, in a non-uniformly growing economy with production-prices, are*

$$\begin{aligned}\hat{\mathbf{v}} &= \mathbf{l} + \hat{\mathbf{v}}\hat{\mathbf{A}} \\ &= \mathbf{l} + \hat{\mathbf{v}}\mathbf{A} + \hat{\mathbf{v}}(\mathbf{A}g + \mathbf{\Gamma}) + \hat{\mathbf{v}}\mathbf{C},\end{aligned} \tag{15}$$

which is the sum of direct, indirect, hyper and super-indirect labour costs.[9]

The hyper and super-integrated coefficients are identical in circumstances of zero non-uniform growth and zero capitalist consumption. And both the hyper and super-integrated coefficients reduce to the classical definition of labour-value in circumstances of zero growth and zero capitalist consumption (i.e., Smith's "early and rude state"). As before the super-integrated subsystems collectively output the real wage and exhaust the total supplied labour; in consequence, $\mathbf{l}\mathbf{q}^{\mathrm{T}} = \hat{\mathbf{v}}\mathbf{w}^{\mathrm{T}}$.

We now state the main result of this paper:

Theorem 2. *The production-prices of a non-uniformly growing economy are proportional to the super-integrated labour coefficients,* $\mathbf{p} = \hat{\mathbf{v}}w$.

Proof. From (14), $\mathbf{pC} = (1/Y)\mathbf{pc}^{\mathrm{T}}\mathbf{p}(\mathbf{A}(\pi - g) - \mathbf{\Gamma})$. Hence, $\mathbf{p}(\mathbf{A}(\pi - g) - \mathbf{\Gamma}) = (Y/\mathbf{pc}^{\mathrm{T}})\mathbf{pC}$. Write price equation (7) in the equivalent form, $\mathbf{p} = \mathbf{pA} + \mathbf{p}(\mathbf{A}g + \mathbf{\Gamma}) + \mathbf{p}(\mathbf{A}(\pi - g) - \mathbf{\Gamma}) + \mathbf{l}w$, and then substitute to yield $\mathbf{p} = \mathbf{pA} + \mathbf{p}(\mathbf{A}g + \mathbf{\Gamma}) + (Y/\mathbf{pc}^{\mathrm{T}})\mathbf{pC} + \mathbf{l}w$. Since capitalists spend what they earn, $\mathbf{pc}^{\mathrm{T}} = Y$. Hence, $\mathbf{p} = \mathbf{pA} + \mathbf{p}(\mathbf{A}g + \mathbf{\Gamma}) + \mathbf{pC} + \mathbf{l}w = \mathbf{p}\hat{\mathbf{A}} + \mathbf{l}w = \mathbf{l}(\mathbf{I} - \hat{\mathbf{A}})^{-1}w = \hat{\mathbf{v}}w$. □

Production-prices, in Pasinetti's non-uniformly growing economy, are the total wage bill of the direct, indirect, hyper and super-indirect labour supplied to reproduce unit commodities. Once again, a more general definition of labour costs recreates Adam Smith's "early and rude state" of society in which 'labour embodied' equals 'labour commanded'.[10]

7. Conclusion: A general solution to Marx's transformation problem

A 'category-mistake' (Ryle, [1949] 1984) is the conceptual mistake of expecting some concept or thing to possess properties it cannot have. The classical labour theory of value suffers from such a category-mistake. Production-prices include the nominal cost but classical labour-values exclude the labour cost of reproducing the capitalist class. The classical labour theory ultimately attempts to identify a commensurate relationship between costs structures defined by incommensurate accounting conventions. Hence Marx's transformation problem, which is an specific manifestation of this category-mistake (Wright, 2014).

Pasinetti recognizes the need to extend the classical theory and construct more general measures of labour cost that take into account additional, non-technical features of the circumstances of production, such as the labour cost of net capital investment, in order

[9]Note that definition 5 reduces to definition 3 in conditions of zero growth.
[10]Note that theorem 2 reduces to theorem 1 in conditions of zero growth.

to explain the structure of 'natural' price systems. Despite this important conceptual advance, Pasinetti nonetheless believes, in virtue of the long-standing problems of the classical labour theory and the mismatch between production-prices and his hyper-integrated labour coefficients, that a labour theory of value "can never reflect the price structure that emerges from the operation of the market in a capitalist economy" (Pasinetti, 1981, p. 153).

Theorem 2 demonstrates the contrary. Production-prices, in both stationary and non-uniformly growing economies, are proportional to physical quantities of labour, specifically the vertically super-integrated labour coefficients, which additionally include the labour supplied to produce net investment and capitalist consumption goods. The transformation problem dissolves once we adopt a more general definition of labour-value that reflects the cost structure that "emerges from the operation of the market in a capitalist economy". The vertically super-integrated labour coefficients are the labour costs that production-prices represent.

Pasinetti's hyper-integrated coefficients, in contrast, reflect the cost structure of hyper-subsystems that correspond to a pre-institutional stage of analysis that lacks specifically capitalist characteristics. Pasinetti has therefore provided us with a "complete generalization of Marx's 'transformation problem"' (Pasinetti, 1988) only by reproducing, at a higher level of generality, the underlying category-mistake that originally caused its manifestation.

Pasinetti's "separation theorem", and the analytical device of partitioning an economic system into 'natural' subsystems, is a powerful analytic device that reveals fundamental constraints and requirements that any economic system must satisfy. Reati (2000) argues, for example, that the labour theory of value is "strongly reinforced by Pasinetti" since the 'natural' stage of analysis removes the distorting effects of capitalist economic relations and demonstrates the analytical relevance of the pure labour theory of value.

However, Pasinetti's specific proposal to restrict the labour theory of value to a normative role, a kind of "logical frame of reference", is unwarranted and unfortunately misrepresents the relation between money and labour in actual economic systems. Pasinetti's theoretical innovations, specifically his proposal to completely generalize Sraffa's concept of a subsystem, instead point in the opposite direction and toward a full generalization of the classical labour theory and its reinstatement as the foundational theory of value for economic analysis within the "production paradigm" (Pasinetti, 1986).

More generally, the post-Sraffian separation of the classical 'surplus' approach to income distribution from its labour theory of value reproduces the classical category-mistake. The separation of the dual systems of natural prices and labour costs does not constitute a sophisticated rejection of naive substance theories of value but indicates a failure to resolve the classical contradictions, such as Marx's transformation problem. The 'surplus approach' therefore dispenses with an essential aim of a theory of economic value, which is to explain what the unit of account might measure or refer to. Theorem 2 starts to put the pieces back together again.

8. Appendix

Here, for clarity, I present numerical examples of Theorems 1 and 2 for a $n = 2$ economy, and a brief discussion of time-varying final demand.

8.1. Theorem 1: A numerical example

The following proposition, which links the technique, capitalist consumption and the profit-rate, will be useful:

Proposition 3. *In a steady-state economy with production prices,* $\text{Tr}(\mathbf{CA}^{-1}) = \pi$.

Proof. Since capitalists spend what they earn, $\mathbf{pAq}^{\text{T}}\pi = \mathbf{pc}^{\text{T}}$. Hence, $1/\mathbf{pAq}^{\text{T}} = \pi/\mathbf{pc}^{\text{T}}$. From equation (11), $\mathbf{CA}^{-1} = (1/\mathbf{pAq}^{\text{T}})\mathbf{c}^{\text{T}}\mathbf{p}$. Substitute for $1/\mathbf{pAq}^{\text{T}}$ to yield, $\mathbf{CA}^{-1} = (\pi/\mathbf{pc}^{\text{T}})\mathbf{c}^{\text{T}}\mathbf{p}$. An inner product is the trace of its outer product, i.e. $\text{Tr}(\mathbf{c}^{\text{T}}\mathbf{p}) = \mathbf{pc}^{\text{T}}$. Hence, $\text{Tr}(\mathbf{CA}^{-1}) = \pi$. □

Observe a stationary economy with (i) technique, $\mathbf{A} = \begin{bmatrix} 0 & 0.5 \\ 0.25 & 0 \end{bmatrix}$ and $\mathbf{l} = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and (ii) capitalist consumption matrix, $\mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0.005 & 0.006 \end{bmatrix}$.

The technique and capitalist consumption matrix determine the super-integrated labour coefficients, i.e. from definition 3, $\hat{\mathbf{v}} = \mathbf{l}(\mathbf{I} - (\mathbf{A} + \mathbf{C}))^{-1} = \begin{bmatrix} 1.74 & 2.89 \end{bmatrix}$.

The technique and capitalist consumption matrix determine the profit-rate, i.e. from Proposition 3, $\pi = \text{Tr}\left(\begin{bmatrix} 0 & 0 \\ 0.012 & 0.02 \end{bmatrix}\right) = 0.02$. The profit-rate determines production-prices, i.e. from equation (7), $\mathbf{p} = \mathbf{l}(\mathbf{I} - \mathbf{A}(1 + \pi))^{-1}w = \begin{bmatrix} 1.74 & 2.89 \end{bmatrix}w$.

Hence, $\mathbf{p} = \hat{\mathbf{v}}w$, as per Theorem 1.

8.2. Theorem 2: A numerical example

The following proposition, which links the technique, non-uniform growth, capitalist consumption and the profit-rate, will be useful:

Proposition 4. *In a non-uniformly growing economy with production prices,*

$$\text{Tr}\left(\mathbf{C}\,(\mathbf{A}(\pi - g) - \mathbf{\Gamma})^{-1}\right) = 1.$$

Proof. Since capitalists spend what they earn, hence $1/Y = 1/\mathbf{pc}^{\text{T}}$. From equation (14), $\mathbf{C}(\mathbf{A}(\pi-g)-\mathbf{\Gamma})^{-1} = (1/Y)\mathbf{c}^{\text{T}}\mathbf{p}$. Substitute for $1/Y$ to yield, $\mathbf{C}(\mathbf{A}(\pi-g)-\mathbf{\Gamma})^{-1} = (1/\mathbf{pc}^{\text{T}})\mathbf{c}^{\text{T}}\mathbf{p}$. An inner product is the trace of its outer product, i.e. $\text{Tr}(\mathbf{c}^{\text{T}}\mathbf{p}) = \mathbf{pc}^{\text{T}}$. Hence, $\text{Tr}(\mathbf{C}(\mathbf{A}(\pi - g) - \mathbf{\Gamma})^{-1}) = 1$. □

Observe a non-uniformly growing economy at time t with (i) technique, $\mathbf{A} = \begin{bmatrix} 0 & 0.5 \\ 0.25 & 0 \end{bmatrix}$ and $\mathbf{l} = \begin{bmatrix} 1 & 2 \end{bmatrix}$, (ii) capitalist consumption matrix, $\mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0.0063 & 0.00088 \end{bmatrix}$, (iii) $g = 0.01$, and (iv) $\mathbf{\Gamma} = \begin{bmatrix} 0.00095 & 0 \\ 0 & 0.0074 \end{bmatrix}$.

This is sufficient information to compute the super-integrated labour-values. From definition 5, $\hat{\mathbf{v}} = \mathbf{l}(\mathbf{I} - \mathbf{A}(1 + g) - \mathbf{\Gamma} - \mathbf{C})^{-1} = \begin{bmatrix} 1.75 & 2.91 \end{bmatrix}$.

From Proposition 4, $\text{Tr}\left(\mathbf{C}(\mathbf{A}(\pi - g) - \mathbf{\Gamma})^{-1}\right) = 1$. Solve to yield the profit-rate, $\pi = 0.037$. Production prices, from equation (7), are then $\mathbf{p} = \mathbf{l}(\mathbf{I}-\mathbf{A}(1+\pi))^{-1}w = \begin{bmatrix} 1.75 & 2.91 \end{bmatrix}w$.

Hence $\mathbf{p} = \hat{\mathbf{v}}w$, as per Theorem 2.

8.3. The trajectory of final demand

Final demand, $\mathbf{n}(t)$, grows exponentially, such that $\mathbf{n}(t) = \mathbf{w}(t) + \mathbf{c}(t) = [w_i(0)\mathrm{e}^{(g+r_i)t}] + [c_i(0)\mathrm{e}^{(g+r_i)t}]$ (see section 6.2). In general, given non-uniform growth, that is $r_i \neq r_j$ for some i and j, then matrices $\mathbf{\Gamma}$ and $\mathbf{C}$ and the profit-rate, π, are not constant but vary with time. In consequence, both production-prices, $\mathbf{p}$, and the vertically super-integrated labour coefficients, $\hat{\mathbf{v}}$, vary along the growth trajectory. Nonetheless, Theorem 2 holds at every instant of time. In consequence, the proportionality of production-prices and super-integrated labour-values is a time-invariant property of Pasinetti's model.

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